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On the Hersch–Payne–Schiffer Inequalities for Steklov Eigenvalues*

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Abstract. We prove that the Hersch–Payne–Schiffer isoperimetric inequality for the nth nonzero Steklov eigenvalue of a bounded simply connected planar domain is sharp for all $n \geqslant 1$. The equality is attained in the limit by a sequence of simply connected domains degenerating into a disjoint union of n identical disks. Similar results are obtained for the product of two consecutive Steklov eigenvalues. We also give a new proof of the Hersch–Payne–Schiffer inequality for $n = 2$ and show that it is strict in this case.

Key words: Steklov eigenvalue problem, eigenvalue, isoperimetric inequality.

1. Introduction and Main Results

1.1. Steklov eigenvalue problem. Let Ω be a simply connected bounded planar domain with Lipschitz boundary, and let $\rho \in L^\infty(\partial\Omega)$ be a nonnegative nonzero function. The *Steklov eigenvalue problem* [28] reads

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial u/\partial\nu = \sigma\rho u & \text{on } \partial\Omega, \end{cases} \tag{1.1.1}$$

where $\partial/\partial\nu$ is the outward normal derivative. There are several physical interpretations of the Steklov problem ([3], [26]). In particular, it describes the vibrations of a free membrane whose entire mass $M(\Omega)$ is distributed on the boundary with density ρ,

$$M(\Omega) = \int_{\partial\Omega} \rho(s)\, ds. \tag{1.1.2}$$

If $\rho \equiv 1$, then the mass of Ω is equal to the length of $\partial\Omega$.

The spectrum of the Steklov problem is discrete, and the eigenvalues

$$0 = \sigma_0 < \sigma_1(\Omega) \leqslant \sigma_2(\Omega) \leqslant \sigma_3(\Omega) \leqslant \cdots \nearrow \infty$$

satisfy the following variational characterization [3, pp. 95, 103]:

$$\sigma_n(\Omega) = \inf_{E_n} \sup_{0 \neq u \in E_n} \frac{\int_\Omega |\nabla u|^2\, dz}{\int_{\partial\Omega} u^2 \rho\, ds}, \qquad n = 1, 2, \dots . \tag{1.1.3}$$

Here the infimum is taken over all n-dimensional subspaces E_n of the Sobolev space $H^1(\Omega)$ that are orthogonal to constants on $\partial\Omega$, i.e., satisfy $\int_{\partial\Omega} u(s)\rho(s)\, ds = 0$ for all $u \in E_n$. Note that, just as in the case of the Neumann boundary conditions, the Steklov spectrum always starts with the eigenvalue $\sigma_0 = 0$, and the corresponding eigenfunctions are constant.

If $\rho \equiv 1$, then the Steklov eigenvalues and eigenfunctions coincide with those of the *Dirichlet-to-Neumann operator*

$$\Gamma\colon H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

defined by

$$\Gamma f = \frac{\partial}{\partial\nu}(\mathscr{H} f),$$

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where $\mathscr{H} f$ is the unique harmonic extension of the function $f \in H^{1/2}(\partial\Omega)$ into the interior of Ω. If the boundary is smooth, then the Dirichlet-to-Neumann operator is a first-order elliptic pseudo-differential operator [30, p. 37–38]. It has various important applications, particularly to the study of inverse problems [31].

1.2. Upper bounds on Steklov eigenvalues. The present paper is motivated by the following

Question 1.2.1. *How large can the nth eigenvalue of the Steklov problem be on a bounded simply connected planar domain of given mass?*

For $n = 1$, the answer to Question 1.2.1 was given in 1954 by Weinstock [33]. He proved that

$$\sigma_1(\Omega)M(\Omega) \leqslant 2\pi \tag{1.2.2}$$

with the equality attained on a disk with $\rho \equiv \text{const}$. Note that the first eigenvalue of the unit disk $\mathbb{D}$ with $\rho \equiv 1$ has multiplicity two, and $\sigma_1(\mathbb{D}) = \sigma_2(\mathbb{D}) = 1$. Various extensions of Weinstock's inequality and related results can be found in [2], [18], [4], [10], and [11]; see also [1, Sec. 8] for a recent survey.

In 1974, Hersch, Payne, and Schiffer [20, p. 102] proved the following estimates:

$$\sigma_m(\Omega)\sigma_n(\Omega)M(\Omega)^2 \leqslant \begin{cases} (m+n-1)^2\pi^2 & \text{if } m+n \text{ is odd,} \\ (m+n)^2\pi^2 & \text{if } m+n \text{ is even.} \end{cases} \tag{1.2.3}$$

In particular, for $m = n$ and $m = n + 1$ we obtain

$$\sigma_n(\Omega)M(\Omega) \leqslant 2\pi n, \qquad n = 1, 2, \dots, \tag{1.2.4}$$

$$\sigma_n(\Omega)\sigma_{n+1}(\Omega)M(\Omega)^2 \leqslant 4\pi^2 n^2, \qquad n = 1, 2, \dots. \tag{1.2.5}$$

1.3. Main results. If $n = 1$, it is easily seen that (1.2.4) and (1.2.5) become equalities on a disk with constant density ρ on the boundary. It was indicated in [20] that the estimates (1.2.3) are not expected to be sharp for all m and n. While this is likely to be true, it turns out that if $m = n$ or $m = n + 1$, then these inequalities *are* sharp for all $n \geqslant 1$.

Theorem 1.3.1. *There exists a family of simply connected bounded Lipschitz domains $\Omega_\varepsilon \subset \mathbb{R}^2$ degenerating into a disjoint union of n identical disks as $\varepsilon \to 0+$ such that, for the Steklov problems with $\rho \equiv 1$ on $\partial\Omega_\varepsilon$ for all ε, one has*

$$\lim_{\varepsilon \to 0+} \sigma_n(\Omega_\varepsilon)M(\Omega_\varepsilon) = 2\pi n, \qquad n = 2, 3, \dots, \tag{1.3.2}$$

and

$$\lim_{\varepsilon \to 0+} \sigma_n(\Omega_\varepsilon)\sigma_{n+1}(\Omega_\varepsilon)M(\Omega_\varepsilon)^2 = 4\pi^2 n^2, \qquad n = 2, 3, \dots. \tag{1.3.3}$$

In particular, the Hersch–Payne–Schiffer inequalities (1.2.4) *and* (1.2.5) *are sharp for all $n \geqslant 1$.*

Remark 1.3.4. As we show in Section 2.2, to obtain (1.3.2) and (1.3.3), one has to be careful in the choice of a family of domains degenerating into a disjoint union of n identical disks.

It would be of interest to check whether each of Eqs. (1.3.2) and (1.3.3) *implies* that the family Ω_ε converges in an appropriate sense to a disjoint union of n identical disks.

Remark 1.3.5. If $\rho \equiv 1$, then the estimate (1.2.4) and the standard isoperimetric inequality in $\mathbb{R}^2$ imply that

$$\sigma_n(\Omega)\sqrt{\text{Area}(\Omega)} < n\sqrt{\pi}, \qquad n \geqslant 2.$$

There is no known sharp "isoareal" estimate on σ_n, $n \geqslant 2$ (see [16, Open problem 25]).

Theorem 1.3.1 gives an almost complete answer to Question 1.2.1. It remains to establish whether inequality (1.2.4) is *strict* for all $n \geqslant 2$. We believe that this is true. A modification of the method introduced in [14] allows one to prove this result for $n = 2$.

Theorem 1.3.6. *Inequality* (1.2.4) *is strict for $n = 2$*:

$$\sigma_2(\Omega)M(\Omega) < 4\pi. \tag{1.3.7}$$

The proof of Theorem 1.3.6 uses the Riemann mapping theorem similarly to [29], [33], and [14]. Note that this approach is very different from the techniques in [20].

1.4. Comparison with the Dirichlet and Neumann cases. To put inequalities (1.2.2) and (1.3.7) into perspective, let us state similar results for the Dirichlet and Neumann eigenvalue problems. Since these eigenvalue problems describe vibrations of a membrane of unit density, it follows that the mass of the membrane is equal to its area.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let $0 < \lambda_1(\Omega) \leqslant \lambda_2(\Omega) \leqslant \dots$ and $0 = \mu_0 < \mu_1(\Omega) \leqslant \mu_2(\Omega) \leqslant \dots$ be the Dirichlet and Neumann eigenvalues of Ω, respectively. Then the following assertions are true.

• *Faber–Krahn inequality*: $\lambda_1(\Omega)\,\mathrm{Area}(\Omega) \geqslant \pi\lambda_1(\mathbb{D})$. (This was conjectured in [27] and proved in [12] and [22], and a weaker version was obtained in [7].)

• *Krahn's inequality*: $\lambda_2(\Omega)\,\mathrm{Area}(\Omega) > 2\pi\lambda_1(\mathbb{D})$ [23]; see [1, p. 110] for an interesting discussion of the history of this result). The equality is attained in the limit by a sequence of domains degenerating into a disjoint union of two identical disks.

• *Szegő–Weinberger inequality*: $\mu_1(\Omega)\,\mathrm{Area}(\Omega) \leqslant \pi\mu_1(\mathbb{D})$. This estimate was proved in [29] for simply connected planar domains. In [32], the result was extended to arbitrary domains in all dimensions.

• If Ω is simply connected, then $\mu_2(\Omega)\,\mathrm{Area}(\Omega) \leqslant 2\pi\mu_1(\mathbb{D})$. This inequality was recently proved in [14]. It is an open question whether it holds for multiply connected planar domains. The equality is attained in the limit by a sequence of domains degenerating into a disjoint union of two identical disks.

For higher Dirichlet and Neumann eigenvalues, no sharp estimates of this type are known, and the situation is quite different from the Steklov case. As was mentioned in [14, Remark 1.2.8], a disjoint union of n identical disks cannot maximize the quantity $\mu_n(\Omega)\,\mathrm{Area}(\Omega)$ for sufficiently large n, because this would contradict Weyl's law. The same argument applies to the minimization problem for $\lambda_n(\Omega)\,\mathrm{Area}(\Omega)$. In fact, it is conjectured that for $n = 3$ the minimizer is a single disk (see [34], [6]). This conjecture is supported by numerical computations.

1.5. Outline of the paper. In Section 2, we prove Theorem 1.3.1. We also construct a family of domains whose Steklov spectrum completely "collapses" to zero in the limit as the domains degenerate into a disjoint union of two unit disks. This phenomenon is quite surprising and occurs for neither Dirichlet nor Neumann eigenvalues. The rest of the paper deals with the proof of Theorem 1.3.6. In Section 3, the "folding and rearrangement" technique, introduced in [25] and developed in [14], is adapted to the Steklov problem. In Section 4, we combine analytic and topological arguments to construct a two-dimensional space of trial functions for the variational characterization (1.1.3) of the second Steklov eigenvalue. This space of trial functions is then used to prove inequality (1.3.7).

2. Maximization and Collapse of Steklov eigenvalues

2.1. Proof of Theorem 1.3.1. Let us start with the case $n = 2$. For each $\varepsilon \in (0, 1/10)$, consider the simply connected planar domain

$$\Omega_\varepsilon = \{|z - 1 + \varepsilon| < 1\} \cup \{|z + 1 - \varepsilon| < 1\} \subset \mathbb{C}. \tag{2.1.1}$$

As $\varepsilon \to 0+$, Ω_ε degenerates into a disjoint union of two identical unit disks.

Lemma 2.1.2. *Let $\rho \equiv 1$ on $\partial\Omega_\varepsilon$ for any ε. Then*

$$\lim_{\varepsilon \to 0+} \sigma_2(\Omega_\varepsilon) = 1.$$

Recall that if $\rho \equiv 1$, then $\sigma_1(\mathbb{D}) = \sigma_2(\mathbb{D}) = 1$.

Remark 2.1.3. While this lemma is not surprising, it does not follow in a straightforward way from general results on convergence of eigenvalues. The difficulty is that the family Ω_ε is not uniformly Lipschitz. Equivalently, the family Ω_ε does not satisfy the uniform cone condition (see

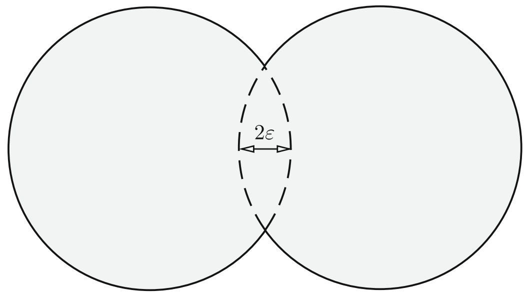

Fig. 1. The domain Ω_ε for $n = 2$

[8, p. 49] or [17, p. 53]). This means that one cannot choose the Lipschitz constant uniformly in *both* $z \in \partial\Omega_\varepsilon$ and ε. Indeed, one can readily see that the Lipschitz constant blows up near $z = 0$ as $\varepsilon \to 0$. In this situation, the Steklov eigenvalues may a priori have a rather surprising limiting behavior; see Section 2.2.

Proof of Lemma 2.1.2. For each $\varepsilon \in (0, 1/10)$,

$$\sigma_2(\Omega_\varepsilon) M(\Omega_\varepsilon) \leqslant 4\pi$$

by (1.2.4). Since $\lim_{\varepsilon\to 0+} M(\Omega_\varepsilon) = 4\pi$, we have

$$\limsup_{\varepsilon\to 0+} \sigma_2(\Omega_\varepsilon) \leqslant 1.$$

It remains to show that

$$\liminf_{\varepsilon\to 0+} \sigma_2(\Omega_\varepsilon) \geqslant 1. \tag{2.1.4}$$

In view of Remark 2.1.3, to apply standard results on convergence of eigenvalues, we need to "desingularize" the family of domains Ω_ε. Let $\Omega'_\varepsilon = \Omega_\varepsilon \cap \{\operatorname{Re} z < 0\}$. Consider the following auxiliary mixed eigenvalue problem on Ω'_ε: we impose the Neumann condition on $\Omega_\varepsilon \cap \{\operatorname{Re} z = 0\}$ and retain the Steklov condition on $\partial\Omega'_\varepsilon \cap \partial\Omega_\varepsilon$. Let $0 = \sigma_0^N(\Omega'_\varepsilon) < \sigma_1^N(\Omega'_\varepsilon) \leqslant \sigma_2^N(\Omega'_\varepsilon) \ldots$ be the eigenvalues of this mixed problem. (It is called a *sloshing* problem; see [13].) Adding the Neumann condition inside the domain increases the space of trial functions and hence, by the standard monotonicity argument [3, p. 100], pushes the eigenvalues down. Therefore,

$$\sigma_2(\Omega_\varepsilon) \geqslant \sigma_1^N(\Omega'_\varepsilon),$$

and hence to prove (2.1.4) it suffices to show that

$$\lim_{\varepsilon\to 0+} \sigma_1^N(\Omega'_\varepsilon) = 1. \tag{2.1.5}$$

The family of domains Ω'_ε converges to $\mathbb{D}$ in the Hausdorff complementary topology (see [5, p. 101]) as $\varepsilon \to 0+$. Moreover, since the domains Ω'_ε are uniformly Lipschitz in both $z \in \partial\Omega'_\varepsilon$ and $\varepsilon \in (0, 1/10)$, it follows that the extension operators $H^1(\Omega_\varepsilon) \to H^1(\mathbb{R}^2)$ are uniformly bounded [5, p. 198], and the norms of the trace operators are uniformly bounded as well [9]. Note also that the Neumann part $\Omega_\varepsilon \cap \{\operatorname{Re} z = 0\}$ of $\partial\Omega'_\varepsilon$ tends to the single point $z = 0$ as $\varepsilon \to 0+$. Therefore, using the Rayleigh quotient for the sloshing problem [13, p. 673]), we obtain

$$\lim_{\varepsilon\to 0+} \sigma_n^N(\Omega'_\varepsilon) = \sigma_n(\mathbb{D}), \qquad n = 1, 2, \ldots,$$

in the same way as in [5, Corollary 7.4.2]. Taking $n = 1$, we arrive at (2.1.5). This completes the proof of the lemma. $\square$

Let us now complete the proof of Theorem 1.3.1. First, it follows from (1.2.5) and the obvious inequality $\sigma_{n+1}(\Omega_\varepsilon) \geqslant \sigma_n(\Omega_\varepsilon)$ that (1.3.2) implies (1.3.3). Therefore, it suffices to prove (1.3.2). For $n = 2$, it follows from Lemma 2.1.2. For $n > 2$, the proof is similar. Define Ω_ε as the union of n disks of radius $1 + \varepsilon$ centered at the points $z = 2k$, $k = 0, 1, \ldots, n - 1$. We make cuts along the vertical lines $\operatorname{Re} z = 2k - 1$, $k = 1, \ldots, n$, and impose the Neumann boundary conditions along these cuts.

We obtain n auxiliary mixed problems. They are of two types: the first and the last disks have just one cut (we denote the corresponding domains by Ω'_ε as before), and the intermediate disks have two cuts each, one on the left and one on the right. (The corresponding domains are denoted by Ω''_ε.) The spectrum of each of these n auxiliary problems starts from the zero eigenvalue. Using the same monotonicity and convergence argument as above, we obtain

$$\sigma_n(\Omega_\varepsilon) \geqslant \min(\sigma_1^N(\Omega'_\varepsilon), \sigma_1^N(\Omega''_\varepsilon))$$

and

$$\lim_{\varepsilon\to 0+} \sigma_1^N(\Omega'_\varepsilon) = \lim_{\varepsilon\to 0+} \sigma_1^N(\Omega''_\varepsilon) = 1.$$

Therefore, $\liminf_{\varepsilon\to 0+} \sigma_n(\Omega_\varepsilon) \geqslant 1$. Since $\lim_{\varepsilon\to 0+} M(\Omega_\varepsilon) = 2\pi n$, it follows from (1.2.4) that $\limsup_{\varepsilon\to 0+} \sigma_n(\Omega_\varepsilon) \leqslant 1$. Hence $\lim_{\varepsilon\to 0+} \sigma_n(\Omega_\varepsilon) = 1$, and this completes the proof of Theorem 1.3.1.

2.2. Collapse of the Steklov spectrum: an example. One could ask why the sequence Ω_ε is constructed by pulling the disks apart rather than by joining them with a tiny passage disappearing as $\varepsilon \to 0$. While this looks geometrically more natural, it turns out that the behavior of the Steklov spectrum under such a degeneration can be quite unexpected.

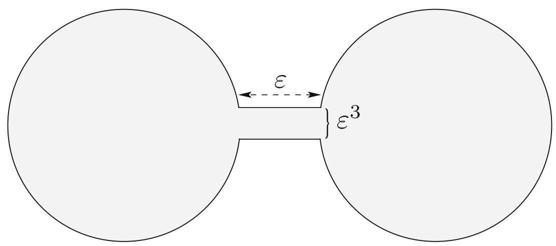

Fig. 2. The domain Σ_ε

As before, set $\rho \equiv 1$. Let $\Sigma_\varepsilon = \mathbb{D}_1 \cup P_\varepsilon \cup \mathbb{D}_2$, where $\mathbb{D}_1$ and $\mathbb{D}_2$ are two copies of the unit disk joined by a rectangular passage P_ε of length ε and width ε^3 (see Figure 2); the shorter sides of P_ε are chords of the boundary circles $\partial\mathbb{D}_1$ and $\partial\mathbb{D}_2$. What is essential in this construction is that the width of the passage tends to zero much faster than its length. For simplicity, we assume that the disks and the passage are chosen in such a way that the domain Σ_ε is symmetric with respect to both coordinate axes. Then, surprisingly enough,

$$\lim_{\varepsilon\to 0+} \sigma_n(\Sigma_\varepsilon) = 0 \tag{2.2.1}$$

for *all* $n = 1, 2, \dots$. To show this, consider pairwise orthogonal trial functions vanishing in the set $(\mathbb{D}_1 \cup \mathbb{D}_2) \setminus P_\varepsilon$ and equal to $\sin(2\pi n x/\varepsilon)$ in the passage P_ε. For each n, the gradient of the trial function is of the order of n/ε, the area of P_ε is ε^4, and the length of the boundary of P_ε is 2ε. Note that the constructed trial functions glue continuously along smaller sides of P_ε and hence belong to the Sobolev space $H^1(\Sigma_\varepsilon)$. Therefore, for each n, the corresponding Rayleigh quotient is of the order of $n^2\varepsilon$ and tends to zero as $\varepsilon \to 0+$. This proves (2.2.1).

Similar constructions were studied in the context of the Neumann boundary conditions (see [21], [15], and references therein). However, the Neumann eigenvalues of Σ_ε converge to the corresponding eigenvalues of the disjoint union of two disks as $\varepsilon \to 0+$. The total "collapse" of the Steklov spectrum in the example above is caused by the fact that the denominator of the Rayleigh quotient is an integral over the *boundary*. Note that the perimeter of the passage P_ε tends to zero much slower than its area, and hence, for every fixed n, the numerator in the Rayleigh quotient vanishes much faster than the denominator.

In the subsequent sections, we prove Theorem 1.3.6.

3. Folding and Rearrangement of Measure

3.1. Conformal mapping into a disk. Let Ω be a simply connected planar domain with Lipschitz boundary. As before, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk. By the Riemann mapping theorem (see [30, p. 342]), there exists a conformal equivalence $\phi\colon \mathbb{D} \to \Omega$ that extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{\Omega}$. (Slightly abusing notation, here and further on we denote a conformal map and its extension to the boundary by the same symbol.) Let ds be the arc length measure on $\partial\Omega$, and let $d\mu$ be the pullback by ϕ of the measure $\rho(s)\,ds$,

$$\int_{\mathscr{O}} d\mu = \int_{\phi(\mathscr{O})} \rho(s)\,ds \tag{3.1.1}$$

for any open set $\mathscr{O} \subset S^1$. Taking into account (3.1.1) and using the conformal invariance of the Dirichlet integral, we rewrite the variational characterization (1.1.3) of σ_2 as follows:

$$\sigma_2(\Omega) = \inf_E \sup_{0\neq u\in E} \frac{\int_{\mathbb{D}} |\nabla u|^2\,dz}{\int_{S^1} u^2\,d\mu}. \tag{3.1.2}$$

Here the infimum is taken over all subspaces $E \subset H^1(\mathbb{D})$ such that $\dim E = 2$ and $\int_{S^1} u\,d\mu = 0$ for all $u \in E$.

3.2. Hyperbolic caps. Let γ be a geodesic in the Poincaré disk model, that is, a diameter or the intersection of the disk with a circle orthogonal to S^1. Each connected component of $\mathbb{D}\setminus\gamma$ is called a *hyperbolic cap* [14]. Given $p \in S^1$ and $l \in (0, 2\pi)$, let $a_{l,p}$ be the hyperbolic cap such that the circular segment $\partial a_{l,p} \cap S^1$ has length l and is centered at p (see Figure 3). This gives an identification of the space $\mathscr{HC}$ of all hyperbolic caps with the cylinder $(0, 2\pi) \times S^1$. Given a cap $a \in \mathscr{HC}$, let $\tau_a\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be the reflection in the hyperbolic geodesic bounding a. That is, τ_a is the unique nontrivial conformal involution of $\mathbb{D}$ leaving every point of the geodesic $\partial a \cap \mathbb{D}$ fixed. In particular, $\tau_a(a) = \mathbb{D}\setminus\overline{a}$.

The *lift* of a function $u\colon \overline{a} \to \mathbb{R}$ is the function $\tilde{u}\colon \overline{\mathbb{D}} \to \mathbb{R}$ defined by

$$\tilde{u}(z) = \begin{cases} u(z) & \text{if } z \in \overline{a}, \\ u(\tau_a z) & \text{if } z \in \overline{\mathbb{D}\setminus a}. \end{cases} \tag{3.2.1}$$

Observe that

$$\int_{S^1} \tilde{u}\,d\mu = \int_{\partial a\cap S^1} u\,d\mu + \int_{\tau_a(\partial a)\cap S^1} u\circ\tau_a\,d\mu = \int_{\partial a\cap S^1} u\,(d\mu + \tau_a^* d\mu). \tag{3.2.2}$$

The measure

$$d\mu_a = \begin{cases} d\mu + \tau_a^* d\mu & \text{on } \partial a \cap S^1, \\ 0 & \text{on } S^1 \setminus \partial a \end{cases} \tag{3.2.3}$$

is called the *folded measure*. Equation (3.2.2) can be rewritten as

$$\int_{S^1} \tilde{u}\,d\mu = \int_{S^1} u\,d\mu_a.$$

3.3. Eigenfunctions on the disk. Given $t \in \mathbb{R}^2$, define $X_t\colon \overline{\mathbb{D}} \to \mathbb{R}$ by $X_t(z) = z\cdot t$, the inner product of z and t in $\mathbb{R}^2$. Let (e_1, e_2) be the standard basis of $\mathbb{R}^2$. Then X_{e_1} and X_{e_2} form a basis of the first Steklov eigenspace on the disk with $\rho \equiv 1$. Using the Hersch renormalization procedure (see [14, Sec. 4.1]), we assume that the center of mass of the measure $d\mu$ is at the origin,

$$\int_{S^1} X_t\,d\mu = 0 \qquad \forall t \in \mathbb{R}^2. \tag{3.3.1}$$

Using a rotation if necessary, we can also assume that

$$\int_{S^1} X_{e_1}^2\,d\mu \geqslant \int_{S^1} X_t^2\,d\mu \qquad \forall t \in S^1. \tag{3.3.2}$$

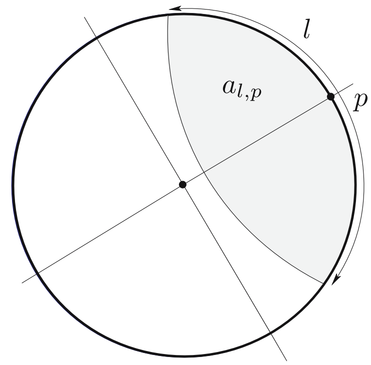

Fig. 3. The hyperbolic cap $a_{l,p}$

3.4. Rearranged measure. Let $a \in \mathscr{HC}$ be a hyperbolic cap, and let $\psi_a \colon \mathbb{D} \to a$ be a conformal equivalence. Following the convention adopted in Section 3.1, we denote its extension $\overline{\mathbb{D}} \to \overline{a}$ again by ψ_a. For each $t \in \mathbb{R}^2$, define $u_a^t \colon \overline{a} \to \mathbb{R}$ by

$$u_a^t(z) = X_t \circ \psi_a^{-1}(z) = t \cdot \psi_a^{-1}(z).$$

The following auxiliary lemma will be used in the proof of Lemma 4.1.1.

Lemma 3.4.1. *The lift of the function u_a^t is not harmonic in $\mathbb{D}$.*

Proof. Suppose that $\tilde{u}_a^t$ is harmonic. Then it is smooth, and the normal derivative of u_a^t vanishes at any point $p \in \partial a \cap \mathbb{D}$ by (3.2.1). It is well known that the vanishing of the normal derivative is preserved by conformal transformations. It follows that the normal derivative of the function $X_t = u_a^t \circ \psi_a$ vanishes on $\psi_a^{-1}(\partial a \cap \mathbb{D}) \subset S^1$. However, a straightforward computation shows that $\partial X_t(s)/\partial n \neq 0$ for any $s \neq \pm t/|t|$. $\square$

Let $w_a^t \in C^\infty(\mathbb{D})$ be the unique harmonic extension of $\tilde{u}_a^t\big|_{S^1}$; that is,

$$\begin{cases} \Delta w_a^t = 0 & \text{in } \mathbb{D}, \\ w_a^t = \tilde{u}_a^t & \text{on } S^1. \end{cases} \tag{3.4.2}$$

These functions will later be used as trial functions in the variational characterization (3.1.2). Observe that

$$\int_{S^1} \tilde{u}_a^t \, d\mu = \int_{S^1} u_a^t \, d\mu_a = \int_{S^1} X_t \psi_a^* \, d\mu_a. \tag{3.4.3}$$

We call the pullback measure

$$d\nu_a = \psi_a^* \, d\mu_a \tag{3.4.4}$$

the *rearranged measure* on S^1.

A family of conformal transformations $\{\psi_a \colon \mathbb{D} \to a\}_{a \in \mathscr{HC}}$ is said to be *continuous* if the map of $(0, 2\pi) \times S^1 \times \mathbb{D}$ into the disk defined by $(l, p, z) \mapsto \psi_{a_{l,p}}(z)$ is continuous. The next lemma describes the properties of the rearranged measure $d\nu_a$ as the cap a degenerates either into the full disk or into a point $p \in S^1$.

Lemma 3.4.5. *There exists a continuous family of conformal equivalences $\{\psi_a \colon \mathbb{D} \to a\}_{a \in \mathscr{HC}}$ such that*

$$\int_{S^1} w_a^t \, d\mu = 0, \tag{3.4.6}$$

$$\lim_{a \to \mathbb{D}} d\nu_a = d\mu, \tag{3.4.7}$$

$$\lim_{a \to p} d\nu_a = R_p^* \, d\mu \tag{3.4.8}$$

for each cap $a \in \mathscr{HC}$ and each $t \in \mathbb{R}^2$, where w_a^t is defined by (3.4.2), *$d\nu_a$ is the rearranged measure given by* (3.4.4), *and $R_p(x) = x - 2(x \cdot p)$ is the reflection in the diameter orthogonal to the vector p.*

A few remarks are in order regarding the last two formulas. As was mentioned in Section 3.2, the space $\mathscr{HC}$ can be identified with the cylinder $(0, 2\pi) \times S^1$, and convergence in $\mathscr{HC}$ is understood in the sense of the usual topology on this cylinder. The topology on measures is induced by the norm

$$\|d\nu\| = \sup_{f \in C(S^1),\, |f| \leqslant 1} \left| \int_{S^1} f\, d\nu \right|. \tag{3.4.9}$$

Proof. Let us give an outline of the proof; for more details, see [14, Sec. 2.5]. Start with any continuous family $\{\phi_a\colon \mathbb{D} \to a\}_{a \in \mathscr{HC}}$ such that $\lim_{a \to \mathbb{D}} \phi_a = \mathrm{id}$. The maps ψ_a are defined by composing the ϕ_a on both sides with automorphisms of the disk occurring in the Hersch renormalization procedure. In particular, (3.4.6) is automatically satisfied. As the cap a converges to the full disk $\mathbb{D}$, the conformal equivalences ψ_a converge to the identity map on $\mathbb{D}$, which implies (3.4.7). Finally, one obtains (3.4.8) by setting $n = 1$ in [14, Lemma 4.3.2]. $\square$

From now on, we fix the family of conformal maps ψ_a defined in Lemma 3.4.5. Lemma 3.4.5 implies that the rearranged measure $d\nu_a$ depends on the cap a continuously. This is essential for the topological argument used in the proof of Proposition 4.2.2.

4. Construction of Trial Functions

4.1. Estimate on the Rayleigh quotient. It follows from (3.4.6) that the functions w_a^t defined by (3.4.2) are admissible in the variational characterization (3.1.2) for σ_2. For each hyperbolic cap $a \in \mathscr{HC}$, consider the two-dimensional space

$$E_a = \{ w_a^t \mid t \in \mathbb{R}^2 \}$$

of trial functions.

Lemma 4.1.1. *For any trial function $w_a^t \in E_a$,*

$$\int_{\mathbb{D}} |\nabla w_a^t|^2\, dz < 2\pi.$$

Proof. It is well known that a harmonic function (such as w_a^t) is the unique minimizer of the Dirichlet energy on the set of all functions in $H^1(\mathbb{D})$ with the same boundary data. By Lemma 3.4.1, the function $\tilde{u}_a^t$ is not harmonic. Since it is continuous, it is not equal to w_a^t in $H^1(\mathbb{D})$. Therefore,

$$\begin{aligned} \int_{\mathbb{D}} |\nabla w_a^t|^2\, dz < \int_{\mathbb{D}} |\nabla \tilde{u}_a^t|^2\, dz &= \int_a |\nabla u_a^t|^2\, dz + \int_{\mathbb{D}\setminus a} |\nabla (u_a^t \circ \tau_a)|^2\, dz \\ &= 2\int_a |\nabla u_a^t|^2\, dz = 2\int_{\mathbb{D}} |\nabla X_t|^2\, dz = 2\underbrace{\sigma_1(\mathbb{D})}_{1} \overbrace{\int_{S^1} X_t^2\, d\theta}^{\pi} = 2\pi. \end{aligned} \tag{4.1.2}$$

where the second and the third equalities follow from the conformal invariance of the Dirichlet energy. $\square$

Let $t_1, t_2 \in S^1$ and $t_1 \cdot t_2 = 0$. Given a hyperbolic cap $a \in \mathscr{HC}$, we have

$$\int_{S^1} (w_a^{t_1})^2\, d\mu = \int_{S^1} (X_{t_1})^2\, d\nu_a \geqslant \frac{1}{2} \int_{S^1} \overbrace{(X_{t_1})^2 + (X_{t_2})^2}^{1}\, d\nu_a = \frac{1}{2} \int_{\partial\Omega} \rho(s)\, ds. \tag{4.1.3}$$

Here the first equality follows from (3.4.2) and (3.4.3), the last equality follows from (3.2.3) and (3.1.1), and we may assume without loss of generality that the inequality in the middle is true. (If it is not, we interchange t_1 and t_2.)

Remark 4.1.4. Since $X_{t_1}^2 + X_{t_2}^2 = 1$ on S^1, we see that the estimate (4.1.3) can be proved as in [19], and it is much easier than the similar result [14, Lemma 2.7.5] for the Neumann problem.

Consider the one-dimensional space

$$V_{t_1} = \{\alpha w_a^{t_1} \mid \alpha \in \mathbb{R}\}$$

of trial functions. It follows from Lemma 4.1.1 and formula (4.1.3) that each function $u \in V_{t_1}$ satisfies

$$\frac{\int_{\mathbb{D}} |\nabla u|^2 \, dz}{\int_{S^1} u^2 \, d\mu} \leqslant \frac{4\pi}{M(\Omega)} \, . \tag{4.1.5}$$

Our next goal is to show that there exists a hyperbolic cap a such that (4.1.5) holds not only for $u \in V_{t_1}$ but for *each* $u \in E_a$. Since E_a is two-dimensional, the estimate (1.3.7) will follow from (4.1.5) and (3.1.2).

4.2. Simple and multiple measures. Given a finite measure $d\nu$ on S^1, consider the quadratic form $V_{d\nu} \colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$V_{d\nu}(t) = \int_{S^1} X_t^2 \, d\nu.$$

Let $\mathbb{R}P^1 = S^1/\mathbb{Z}_2$ be the projective line. We denote by $[t] \in \mathbb{R}P^1$ the element corresponding to the pair of points $\pm t \in S^1$. We say that $[t] \in \mathbb{R}P^1$ is a *maximizing direction* for the measure $d\nu$ if $V_{d\nu}([t]) \geqslant V_{d\nu}([s])$ for any $[s] \in \mathbb{R}P^1$. The measure $d\nu$ is said to be *simple* if there exists a unique maximizing direction. Otherwise, the measure $d\nu$ is said to be *multiple*.

Lemma 4.2.1. *A measure $d\nu$ is multiple if and only if $V_{d\nu}(t)$ does not depend on $t \in S^1$.*

Proof. The lemma follows from the fact that $V_{d\nu}(t)$ is a quadratic form and can be proved by analogy with [14, Lemma 2.6.1]. □

Note that $[e_1]$ is a maximizing direction for the measure $d\mu$ by (3.3.2).

Proposition 4.2.2. *If the measure $d\mu$ is simple, then there exists a cap $a \in \mathscr{HC}$ such that the rearranged measure $d\nu_a$ is multiple.*

The proof is by contradiction. Assume that the measure $d\mu$, as well as the measures $d\nu_a$ for all $a \in \mathscr{HC}$, is simple. Given a hyperbolic cap a, let $[m(a)] \in \mathbb{R}P^1$ be the unique maximizing direction for $d\nu_a$.

By construction, the folded measures $d\mu_a$ depend on the cap a continuously. The family ψ_a is continuous by Lemma 3.4.5, and hence the rearranged measures $d\nu_a$ continuously depend on a. Therefore, the functions $V_{d\nu_a}$ and the unique maximizing direction $[m(a)]$ depend on a continuously as well.

Let us understand the behavior of the maximizing direction as the cap a degenerates either into the full disk or into a point.

Lemma 4.2.3. *Let the measure $d\mu$, as well as the measures $d\nu_a$ for all $a \in \mathscr{HC}$, be simple. Then*

$$\lim_{a \to \mathbb{D}} [m(a)] = [e_1], \tag{4.2.4}$$

$$\lim_{a \to e^{i\theta}} [m(a)] = [e^{2i\theta}]. \tag{4.2.5}$$

Proof. First, note that formula (4.2.4) readily follows from (3.4.7) and (3.3.2). Let us prove (4.2.5). Set $p = e^{i\theta}$. Formula (3.4.8) implies that

$$\lim_{a \to p} \int_{S^1} X_t^2 \, d\nu_a = \int_{S^1} X_t^2 R_p^* \, d\mu = \int_{S^1} X_t^2 \circ R_p \, d\mu = \int_{S^1} X_{R_p t}^2 \, d\mu. \tag{4.2.6}$$

Since $d\mu$ is simple, it follows that $[e_1]$ is the unique maximizing direction for $d\mu$ and the right-hand side of (4.2.6) is maximal for $R_p t = \pm e_1$. By applying R_p on both sides, we obtain $t = \pm e^{2i\theta}$ and hence $[m(a)] = [e^{2i\theta}]$. □

Proof of Proposition 4.2.2. Suppose that for each hyperbolic cap $a \in \mathscr{HC}$ the measure $d\nu_a$ is simple. Recall that the space $\mathscr{HC}$ is identified with the open cylinder $(0, 2\pi) \times S^1$. Define $h\colon (0, 2\pi) \times S^1 \to \mathbb{R}P^1$ by $h(l, p) = [m(a_{l,p})]$. As was mentioned above, the maximizing direction continuously depends on the cap a. Therefore, it follows from Lemma 4.2.3 that h extends to a continuous map on the closed cylinder $[0, 2\pi] \times S^1$ such that

$$h(0, e^{i\theta}) = [e_1], \qquad h(2\pi, e^{i\theta}) = [e^{2i\theta}].$$

This means that h is a homotopy between a trivial loop and a noncontractible loop on $\mathbb{R}P^1$. This is a contradiction. $\square$

4.3. Proof of Theorem 1.3.6. Assume that the measure $d\mu$ is simple. By Proposition 4.2.2, there exists a cap $a \in \mathscr{HC}$ such that the measure $d\nu_a$ is multiple, so that inequality (4.1.5) holds for every $u \in E_a$. Theorem 1.3.6 then readily follows from the variational characterization (3.1.2) of σ_2.

Now suppose that the measure $d\mu$ is multiple. In this case, the proof is easier. Indeed, it follows from Lemma 4.2.1 that every direction $[s] \in \mathbb{R}P^1$ is maximizing for $d\mu$, so that we can use the space

$$E = \{X_t \mid t \in \mathbb{R}^2\}$$

of trial functions in the variational characterization (3.1.2) of σ_2. Replacing w_a^t by X_t and inspecting (4.1.2), we notice that the factor 2 disappears. Therefore, (3.1.2) implies that

$$\sigma_2(\Omega)M(\Omega) \leqslant 2\pi, \tag{4.3.1}$$

which is an even better bound than (1.3.7). This completes the proof of Theorem 1.3.6.

Remark 4.3.2. If the measure $d\mu$ is multiple, then we do not use formula (3.2.1), and hence Lemma 3.4.1 does not apply. Therefore, inequality (4.3.1) is not strict in this case. Indeed, the equality is attained on a disk with $\rho \equiv \text{const}$.

One can readily show that if the domain Ω is symmetric of order $q \geqslant 3$ in the sense of [2] and [3, pp. 136–140]) (for instance, if Ω is a regular q-gon), then the measure $d\mu$ is multiple, provided that the density ρ satisfies the same symmetry condition. Under these assumptions, Eq. (4.3.1) is a special case of [3, Theorem 3.15]. In fact, one can show using Courant's nodal domain theorem for Steklov eigenfunctions [24, Sec. 3] that if the domain Ω and the density ρ are symmetric of order q, then $\sigma_1 = \sigma_2$, so that (4.3.1) is just a consequence of (1.2.2). Indeed, in this case Ω has at least two axes of symmetry, and each of them is a nodal line of an eigenfunction corresponding to σ_1. Therefore, $\operatorname{mult}(\sigma_1) \geqslant 2$. We are not aware of any examples for which the measure $d\mu$ is multiple but the eigenvalue σ_1 is simple.

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References

[1] M. S. Ashbaugh and R. D. Benguria, "Isoperimetric inequalities for eigenvalues of the Laplacian," in: Spectral Theory and Mathematical Physics: a Festschrift in Honor of Barry Simon's 60th Birthday, Proc. Sympos. Pure Math., vol. 76, Part 1, Amer. Math. Soc., Providence, RI, 2007, 105–139.

[2] C. Bandle, "Über des Stekloffsche Eigenwertproblem: Isoperimetrische Ungleichungen für symmetrische Gebiete," Z. Angew. Math. Phys., **19** (1968), 627–237.

[3] C. Bandle, Isoperimetric Inequalities and Applications, Pitman, Boston, 1980.

[4] F. Brock, "An isoperimetric inequality for eigenvalues of the Stekloff problem," Z. Angew. Math. Mech., **81**:1 (2001), 69–71.

[5] D. Bucur and G. Buttazzo, Variational Methods in Shape Optimization Problems, Birkhäuser, Boston, MA, 2005.

[6] D. Bucur and A. Henrot, "Minimization of the third eigenvalue of the Dirichlet Laplacian," Proc. Roy. Soc. London, Ser. A, **456**:1996 (2000), 985–996.
[7] R. Courant, "Beweis des Satzes, daß von allen homogenen Membranen gegebenen Umfanges und gegebener Spannung die kreisförmige den tiefsten Grundton besitzt," Math. Z., **1**:2–3 (1918), 321–328.
[8] M. Delfour and J.-P. Zolěsio, Shapes and Geometries. Analysis, Differential Calculus, and Optimization, Advances in Design and Control, vol. 4, SIAM, Philadelphia, 2001.
[9] Z. Ding, "A proof of the trace theorem of Sobolev spaces on Lipschitz domains," Proc. Amer. Math. Soc., **124**:2 (1996), 591–600.
[10] B. Dittmar, "Sums of reciprocal Stekloff eigenvalues," Math. Nachr., **268** (2004), 44–49.
[11] J. Edward, "An inequality for Steklov eigenvalues for planar domains," Z. Angew. Math. Phys., **45**:3 (1994), 493–496.
[12] G. Faber, "Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt," Sitzungberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München Jahrgang, 1923, 169–172.
[13] D. W. Fox and J. P. Kuttler, "Sloshing frequencies," Z. Angew. Math. Phys., **34**:5 (1983), 668–696.
[14] A. Girouard, N. Nadirashvili, and I. Polterovich, "Maximization of the second positive Neumann eigenvalue for planar domains," J. Differential Geom., **83**:3 (2009), 637–662.
[15] R. Hempel, L. Seco, and B. Simon, "The essential spectrum of Neumann Laplacians on some bounded singular domains," J. Funct. Anal., **102**:2 (1991), 448–483.
[16] A. Henrot, Extremum problems for eigenvalues of elliptic operators, Birkhäuser, Basel, 2006.
[17] A. Henrot and M. Pierre, Variation et optimisation de formes, Springer-Verlag, Berlin, 2005.
[18] A. Henrot, G. Philippin, and A. Safoui, Some isoperimetric inequalities with application to the Stekloff problem, `http://arxiv.org/abs/0803.4242`.
[19] J. Hersch, "Quatre propriétés isopérimétriques de membranes sphériques homogènes," C. R. Acad. Sci. Paris Sér. A-B, **270** (1970), A1645–A1648.
[20] J. Hersch, L. E. Payne, and M. M. Schiffer, "Some inequalities for Stekloff eigenvalues," Arch. Rat. Mech. Anal., **57** (1974), 99–114.
[21] S. Jimbo and Y. Morita, "Remarks on the behavior of certain eigenvalues on a singularly perturbed domain with several thin channels," Comm. Partial Differential Equations, **17**:3–4 (1992), 523–552.
[22] E. Krahn, "Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises," Math. Ann., **94**:1 (1925), 97–100.
[23] E. Krahn, "Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen," Acta Comm. Unic. Dorpat, **A9** (1926), 1–44.
[24] J. R. Kuttler and V. G. Sigillito, "An inequality of a Stekloff eigenvalue by the method of defect," Proc. Amer. Math. Soc., **20** (1969), 357–360.
[25] N. Nadirashvili, "Isoperimetric inequality for the second eigenvalue of a sphere," J. Differential Geom., **61**:2 (2002), 335–340.
[26] L. Payne, "Isoperimetric inequalities and their applications," SIAM Rev., **9**:3 (1967), 453–488.
[27] J. W. S. Rayleigh, The Theory of Sound, vol. 1, McMillan, London, 1877.
[28] W. Stekloff, "Sur les problèmes fondamentaux de la physique mathématique," Ann. Sci. Ecole Norm. Sup., **19** (1902), 455–490.
[29] G. Szegő, "Inequalities for certain eigenvalues of a membrane of given area," J. Rational Mech. Anal., **3** (1954), 343–356.
[30] M. E. Taylor, Partial Differential Equations II. Qualitative Studies of Linear Equations, Applied Mathematical Sciences, vol. 116, Springer-Verlag, New York, 1996.
[31] G. Uhlmann and J. Sylvester, "The Dirichlet to Neumann map and applications," in: Inverse Problems in Partial Differential Equations (Arcata, CA, 1989), SIAM, Philadelphia, PA, 1990, 101–139.

[32] H. F. Weinberger, "An isoperimetric inequality for the N-dimensional free membrane problem," J. Rational Mech. Anal., **5** (1956), 633–636.
[33] R. Weinstock, "Inequalities for a classical eigenvalue problem," J. Rational Mech. Anal., **3** (1954), 745–753.
[34] A. Wolf and J. Keller, "Range of the first two eigenvalues of the Laplacian," Proc. Roy. Soc. London, Ser. A, **447** (1994), 397–412.

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